

FINITENESS OF IRREDUCIBLE HOLOMORPHIC ETA QUOTIENTS OF A GIVEN LEVEL

SOUMYA BHATTACHARYA

ABSTRACT. We show that the number of holomorphic eta quotients of level N , none of which is a product of two holomorphic eta quotients other than 1 and itself is less than $N^2 + 2^n$, where n is the number of distinct prime divisors of N . In particular, the weight of any such eta quotient is at most $\varphi(N_0)^2 d(N)/2$, where φ is Euler's totient function, d is the divisor counting function and N_0 is the product of the distinct prime divisors of N . The finiteness result above is an analog of Zagier's conjecture / Mersmann's theorem which states that: Of any given weight, there are only finitely many irreducible holomorphic eta quotients, none of which is an integral rescaling of another eta quotient. We also show that there exist irreducible holomorphic eta quotients of arbitrarily large weights.

1. INTRODUCTION

The Dedekind eta function is defined by the infinite product:

$$(1.1) \quad \eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \text{ for all } z \in \mathfrak{H},$$

where $q^r = q^r(z) := e^{2\pi i r z}$ for all r and $\mathfrak{H} := \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$. Eta is a holomorphic function on \mathfrak{H} with no zeros. This function has its significance in Number Theory. For example, $1/\eta$ is the generating function for the ordinary partition function $p : \mathbb{N} \rightarrow \mathbb{N}$ (see [1]) and the constant term in the Laurent expansion at 1 of the Epstein zeta function ζ_Q attached to a positive definite quadratic form Q is related via the Kronecker limit formula to the value of η at the root of the associated quadratic polynomial in \mathfrak{H} (see [9]). The value of η at such a quadratic irrationality of discriminant $-D$ is also related via the Lerch/Chowla-Selberg formula to the values of the Gamma function with arguments in $D^{-1}\mathbb{N}$ (see [22]). In fact, the eta function comes up naturally in many other areas of Mathematics (see the Introduction in [4] for a brief overview of them).

The function η is a modular form of weight $1/2$ with a multiplier system on $\text{SL}_2(\mathbb{Z})$ (see [12]). An eta quotient f is a finite product of the form

$$(1.2) \quad \prod \eta_d^{X_d},$$

where $d \in \mathbb{N}$, η_d is the rescaling of η by d , defined by

$$(1.3) \quad \eta_d(z) := \eta(dz) \text{ for all } z \in \mathfrak{H}$$

2010 *Mathematics Subject Classification.* Primary 11F20, 11F37, 11F11; Secondary 11G16, 11F12.

and $X_d \in \mathbb{Z}$. Eta quotients naturally inherit modularity from η : The eta quotient f in (1.2) transforms like a modular form of weight $\frac{1}{2} \sum_d X_d$ with a multiplier system on suitable congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$: The largest among these subgroups is

$$(1.4) \quad \Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\},$$

where

$$(1.5) \quad N := \mathrm{lcm}\{d \in \mathbb{N} \mid X_d \neq 0\}.$$

We call N the *level* of f . Since η is non-zero on \mathfrak{H} , the eta quotient f is holomorphic if and only if f does not have any pole at the cusps of $\Gamma_0(N)$.

An *eta quotient on $\Gamma_0(M)$* is an eta quotient whose level divides M . Let f , g and h be nonconstant holomorphic eta quotients on $\Gamma_0(M)$ such that $f = g \times h$. Then we say that f is *factorizable on $\Gamma_0(M)$* . We call a holomorphic eta quotient f of level N *quasi-irreducible* (resp. *irreducible*), if it is not factorizable on $\Gamma_0(N)$ (resp. on $\Gamma_0(M)$ for all multiples M of N). Here, it is worth mentioning that the notions of irreducibility and quasi-irreducibility of holomorphic eta quotients are conjecturally equivalent (see [4]).

Irreducible holomorphic eta quotients were first considered by Zagier, who conjectured (see [23]) that: *There are only finitely many primitive and irreducible holomorphic eta quotients of a given weight.* An eta quotient f is *primitive* if no eta quotient h and no integer $\nu > 1$ satisfy the equation $f = h_\nu$, where $h_\nu(z) := h(\nu z)$ for all $z \in \mathfrak{H}$. Zagier's conjecture was established by his student Mersmann in an excellent *Diplomarbeit* [14]. In his thesis, Mersmann also proved another conjecture by Zagier on the exhaustiveness of his list of holomorphic eta quotients of weight $1/2$ none of which are integral rescalings of some other eta quotients (see [23]). I gave simplified and short proofs of Mersmann's theorems in [5] and [6]. Since the only holomorphic eta quotients of level 1 are the powers of eta, η is the only irreducible holomorphic eta quotient of level 1. Since the weight of any holomorphic eta quotient is at least $1/2$, each eta quotient of weight $1/2$ is irreducible. In particular, for any $p \in \mathbb{N}$, the eta quotient η_p is irreducible. Again, from Corollary 2 in [4], we know that for any prime p , the holomorphic eta quotients η^p/η_p and η_p^p/η are irreducible (the irreducibility of the later also follows from Lemma 4 below). It is easy to show that any other holomorphic eta quotient of level p except these three is factorizable on $\Gamma_0(p)$. So, the above three are the only irreducible holomorphic eta quotients of a prime level p . Here, we shall show that the finiteness of irreducible holomorphic eta quotients of a given level also holds in general. This in particular, implies that the maximum of the weights of the irreducible holomorphic eta quotients of level N is bounded above with respect to N . Conversely, since the valence formula implies that there are only finitely many holomorphic eta quotients of a given level and weight (see (3.11)), the finiteness of irreducible holomorphic eta quotients of a given level is also implied by such an upper bound.

The finiteness of quasi-irreducible holomorphic eta quotients of a given level (see Theorem 1), has an application in [4], in showing that the levels

of the factors of a holomorphic eta quotient f are bounded above in terms of the level of f .

Before ending this section, let us compare the situation with that of the modular forms (with the trivial multiplier system). Note that the notions of irreducibility and factorizability also makes sense if we replace “holomorphic eta quotients” with “modular forms” above. For example, the modular form $\Delta := \eta^{24}$ of level 1 is not factorizable on $\mathrm{SL}_2(\mathbb{Z})$ since it is a cusp form of the least possible weight on $\mathrm{SL}_2(\mathbb{Z})$. However, Δ is factorizable on $\Gamma_0(2)$:

$$(1.6) \quad \Delta = \eta^8 \eta_2^8 \times \frac{\eta^{16}}{\eta_2^8}.$$

From (3.6), it follows readily that η^p/η_p is holomorphic for each prime p . In particular, so is the rightmost eta quotient in (1.6). Also, from Newman’s criteria (see [16], [17] or [19]), it follows that the multiplier systems of both of the eta quotients on the right hand side of (1.6) are trivial.

For $k \in 2\mathbb{N}$, we define the Eisenstein series E_k by

$$(1.7) \quad E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where the function $\sigma_{k-1} : \mathbb{N} \rightarrow \mathbb{N}$ is given by

$$(1.8) \quad \sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$$

and the k -th Bernoulli number B_k is defined by

$$(1.9) \quad \frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} \cdot t^k.$$

For each even integer $k > 2$, E_k is a modular form of weight k on $\mathrm{SL}_2(\mathbb{Z})$ (see [23]). Since there are no nonzero modular forms of odd weight or weight 2 on $\mathrm{SL}_2(\mathbb{Z})$, neither E_4 nor E_6 is factorizable on $\mathrm{SL}_2(\mathbb{Z})$. However, since $E_6(i) = 0$ and $E_4(e^{2\pi i/3}) = 0$ and since the valence formula (3.8) for $\Gamma_0(2)$ (resp. for $\Gamma_0(4)$) implies that the modular form

$$(1.10) \quad f_1 := \eta_2(i)^{24} \frac{\eta^{16}}{\eta_2^8} - \eta(i)^{24} \frac{\eta_2^{16}}{\eta^8}, \quad \text{resp.} \quad f_2 := \eta_4(e^{2\pi i/3})^8 \frac{\eta^8}{\eta_2^4} - \eta(e^{2\pi i/3})^8 \frac{\eta_4^8}{\eta_2^4}$$

of weight 4 on $\Gamma_0(2)$ (resp. of weight 2 on $\Gamma_0(4)$) only has a simple zero at i in $\Gamma_0(2) \backslash \mathfrak{H}$ (resp. at $e^{2\pi i/3}$ in $\Gamma_0(4) \backslash \mathfrak{H}$), it follows that f_1 is a nontrivial factor of E_6 on $\Gamma_0(2)$ (resp. f_2 is a nontrivial factor of E_4 on $\Gamma_0(4)$). It is easy to check that for all integers $N > 1$, the stabilizers of i and $e^{2\pi i/3}$ in $\Gamma_0(N)$ are trivial. The holomorphy of the eta quotients in the above linear combinations follows trivially from (3.14), once one notes the outermost columns of the matrix in (3.16). The triviality of the multiplier systems of these eta quotients follows again from Newman’s criteria (see [16], [17] or [19]).

Also, it follows from the valence formula (3.8) for $\mathrm{SL}_2(\mathbb{Z})$ that every modular form on $\mathrm{SL}_2(\mathbb{Z})$ has a unique factorization of the form:

$$(1.11) \quad C_0 E_4^a E_6^b \Delta^c \prod_{t \in \mathbb{C} \setminus \{0,1\}} (tE_6^2 + (1-t)E_4^3)^{nt},$$

for some $C_0 \in \mathbb{C}$ and some nonnegative integers a, b, c, n_t , where n_t is zero for all but finitely many t . In particular, any modular form of weight greater than 12 on $\mathrm{SL}_2(\mathbb{Z})$ is factorizable on $\mathrm{SL}_2(\mathbb{Z})$. Clearly, for all $t \in \mathbb{C}$, $tE_6^2 + (1-t)E_4^3$ is nonzero at ∞ . Hence, the valence formula (3.8) for $\mathrm{SL}_2(\mathbb{Z})$ implies that for each t , there exists a $z_t \in \mathfrak{H}$ such that $tE_6(z_t)^2 + (1-t)E_4(z_t)^3 = 0$. If $t \notin \{0, 1\}$, then the stabilizer of z_t in $\mathrm{SL}_2(\mathbb{Z})$ (hence, also in $\Gamma_0(2)$) is trivial. It follows again from the valence formula (3.8) that for all $t \notin \{0, 1\}$, the modular form $tE_6^2 + (1-t)E_4^3$ is not factorizable on $\mathrm{SL}_2(\mathbb{Z})$. Now, the valence formula (3.8) for $\Gamma_0(2)$ implies that for $t \in \mathbb{C} \setminus \{0, 1\}$, the modular form

$$(1.12) \quad \eta_2(z_t)^{24} \frac{\eta^{16}}{\eta_2^8} - \eta(z_t)^{24} \frac{\eta_2^{16}}{\eta^8}$$

of weight 4 on $\Gamma_0(2)$ only has a simple zero at z_t in $\Gamma_0(2) \backslash \mathfrak{H}$. It follows that for $t \in \mathbb{C} \setminus \{0, 1\}$, the modular form above is a factor of $tE_6^2 + (1-t)E_4^3$ on $\Gamma_0(2)$. Thus, every modular form on $\mathrm{SL}_2(\mathbb{Z})$ is factorizable on $\Gamma_0(2) \cap \Gamma_0(4) = \Gamma_0(4)$. However, since the smallest weight of which nonzero modular forms (with the trivial multiplier system) exist is 2, the modular form of weight 2 on $\Gamma_0(N)$ defined by

$$(1.13) \quad NE_2(Nz) - E_2(z)$$

(see [10]) is irreducible for all $N > 1$. In particular, since with each element $x_0 \in \Gamma_0(4) \backslash (\mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q}))$, we can associate a unique modular form of weight 2 on $\Gamma_0(4)$ which vanishes only at x_0 and nowhere else, every modular form of weight greater than 2 on $\Gamma_0(4)$ is factorizable on $\Gamma_0(4)$ (see [4]).

On the contrary, it follows from Theorem 3 below (or from Corollary 2 in [4]) that there exist irreducible holomorphic eta quotients of arbitrarily large weights. We shall also see some irreducibility criteria for holomorphic eta quotients in [7].

2. THE RESULTS

We require to introduce some notations in order to state our main results. For a divisor d of N , we say that d *exactly divides* N and write $d \parallel N$ if $\gcd(d, N/d) = 1$. We define the function $\kappa : \mathbb{N} \rightarrow \mathbb{N}$ by

$$(2.1) \quad \kappa(N) = \varphi(\mathrm{rad}(N)) \prod_{\substack{p \in \wp_N \\ p^n \parallel N}} ((n-1)(p-1) + 2),$$

where φ denotes Euler's totient function, $\mathrm{rad}(N)$ denotes the product of the distinct prime divisors of N and \wp_N denotes the set of prime divisors of N . Let $\psi : \mathbb{N} \rightarrow \mathbb{N}$ denote Dedekind's ψ function:

$$(2.2) \quad \psi(N) := N \prod_{p \in \wp_N} \left(1 + \frac{1}{p}\right).$$

For $N \in \mathbb{N}$, let $d(N)$ (resp. $\omega(N)$) denote the number of divisors (resp. the number of prime divisors) of N . We say that a holomorphic eta quotient f is *divisible* by a holomorphic eta quotient g if f/g is holomorphic. We shall show that

Theorem 1. *For all integers $N > 1$, the following assertions hold:*

- (a) *The weight of any holomorphic eta quotient on $\Gamma_0(N)$ which is not factorizable on $\Gamma_0(N)$ is less than $\kappa(N)/2$.*
- (b) *The number of nonconstant holomorphic eta quotients on $\Gamma_0(N)$ which are not factorizable on $\Gamma_0(N)$ is less than $\varphi(N)\psi(N) + d(N)$.*
- (c) *There are less than $\varphi(N)\psi(N) + 2^{\omega(N)}$ quasi-irreducible holomorphic eta quotients of level N .*

Though the bounds in parts (b) and (c) of the above theorem could be made more precise, here we content ourselves with the above bounds as they quantify the finiteness anyway. In particular, since any irreducible holomorphic eta quotient is quasi-irreducible, from the above theorem, we conclude:

Corollary 1. *For all integers $N > 1$, the following assertions hold:*

- (a) *The weight of any irreducible holomorphic eta quotient of level N is less than $\kappa(N)/2$.*
- (b) *The number of irreducible holomorphic eta quotients of level N is less than $\varphi(N)\psi(N) + 2^{\omega(N)}$.*

In fact, $\kappa(N)/2$ is the smallest possible weight for an eta quotient f such that for each holomorphic eta quotient g which is not factorizable on $\Gamma_0(N)$, f/g is holomorphic:

Theorem 2. *For all $N \in \mathbb{N}$, there exists a holomorphic eta quotient F_N of weight $\kappa(N)/2$ on $\Gamma_0(N)$ such that a holomorphic eta quotient h on $\Gamma_0(N)$ is divisible by F_N if and only if h is divisible by all the holomorphic eta quotients on $\Gamma_0(N)$ which are not factorizable on $\Gamma_0(N)$.*

In the above theorem, the uniqueness of the eta quotient F_N readily follows from the claim. We shall see F_N explicitly in (4.2). We recall that the Reducibility Conjecture (see Conjecture 1' in [4]) states: *Every quasi-irreducible holomorphic eta quotient is irreducible.* Since holomorphic eta quotients on $\Gamma_0(N)$ which are not factorizable on $\Gamma_0(N)$ are in particular quasi-irreducible, it follows from the above theorem that

Corollary 2. *If the Reducibility Conjecture (Conjecture 1' in [4]) holds, then for all $N \in \mathbb{N}$, there exists a holomorphic eta quotient F_N of weight $\kappa(N)/2$ on $\Gamma_0(N)$ such that a holomorphic eta quotient h on $\Gamma_0(N)$ is divisible by F_N if and only if h is divisible by all the irreducible holomorphic eta quotients on $\Gamma_0(N)$.*

We shall also show that

Theorem 3. *For $N \in \mathbb{N}$ and for any divisor t of $N/\text{rad}(N)$, there is an irreducible holomorphic eta quotient of weight*

$$\frac{1}{2}\varphi(\text{rad}(N))\varphi(\text{rad}(\gcd(t, N/t)))$$

on $\Gamma_0(N)$. In particular, for $t = N/\text{rad}(N)$, there exists an irreducible holomorphic eta quotient of level N and of the weight as above.

3. NOTATIONS AND THE BASIC FACTS

By \mathbb{N} we denote the set of positive integers. For $N \in \mathbb{N}$, by \mathcal{D}_N we denote the set of divisors of N . For $X \in \mathbb{Z}^{\mathcal{D}_N}$, we define the eta quotient η^X by

$$(3.1) \quad \eta^X := \prod_{d \in \mathcal{D}_N} \eta_d^{X_d},$$

where X_d is the value of X at $d \in \mathcal{D}_N$ whereas η_d denotes the rescaling of η by d . Clearly, the level of η^X divides N . In other words, η^X transforms like a modular form on $\Gamma_0(N)$. We define the summatory function $\sigma : \mathbb{Z}^{\mathcal{D}_N} \rightarrow \mathbb{Z}$ by

$$(3.2) \quad \sigma(X) := \sum_{d \in \mathcal{D}_N} X_d.$$

Since η is of weight $1/2$, the weight of η^X is $\sigma(X)/2$ for all $X \in \mathbb{Z}^{\mathcal{D}_N}$.

Recall that an eta quotient f on $\Gamma_0(N)$ is holomorphic if it does not have any poles at the cusps of $\Gamma_0(N)$. Under the action of $\Gamma_0(N)$ on $\mathbb{P}^1(\mathbb{Q})$ by Möbius transformation, for $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$, we have

$$(3.3) \quad [a : b] \sim_{\Gamma_0(N)} [a' : \gcd(N, b)]$$

for some $a' \in \mathbb{Z}$ which is coprime to $\gcd(N, b)$ (see [10]). We identify $\mathbb{P}^1(\mathbb{Q})$ with $\mathbb{Q} \cup \{\infty\}$ via the canonical bijection that maps $[\alpha : \lambda]$ to α/λ if $\lambda \neq 0$ and to ∞ if $\lambda = 0$. For $s \in \mathbb{Q} \cup \{\infty\}$ and a weakly holomorphic modular form f on $\Gamma_0(N)$, the order of f at the cusp s of $\Gamma_0(N)$ is the exponent of q^{1/w_s} occurring with the first nonzero coefficient in the q -expansion of f at the cusp s , where w_s is the width of the cusp s (see [10], [18]). The following is a minimal set of representatives of the cusps of $\Gamma_0(N)$ (see [10], [13]):

$$(3.4) \quad \mathcal{S}_N := \left\{ \frac{a}{t} \in \mathbb{Q} \mid t \in \mathcal{D}_N, a \in \mathbb{Z}, \gcd(a, t) = 1 \right\} / \sim,$$

where $\frac{a}{t} \sim \frac{b}{t}$ if and only if $a \equiv b \pmod{\gcd(t, N/t)}$. For $d \in \mathcal{D}_N$ and for $s = \frac{a}{t} \in \mathcal{S}_N$ with $\gcd(a, t) = 1$, we have

$$(3.5) \quad \text{ord}_s(\eta_d; \Gamma_0(N)) = \frac{N \cdot \gcd(d, t)^2}{24 \cdot d \cdot \gcd(t^2, N)} \in \frac{1}{24} \mathbb{N}$$

(see [13]). It is easy to check the above inclusion when N is a prime power. The general case follows by multiplicativity (see (3.12) and (3.15)). It follows that for all $X \in \mathbb{Z}^{\mathcal{D}_N}$, we have

$$(3.6) \quad \text{ord}_s(\eta^X; \Gamma_0(N)) = \frac{1}{24} \sum_{d \in \mathcal{D}_N} \frac{N \cdot \gcd(d, t)^2}{d \cdot \gcd(t^2, N)} X_d.$$

In particular, that implies

$$(3.7) \quad \text{ord}_{a/t}(\eta^X; \Gamma_0(N)) = \text{ord}_{1/t}(\eta^X; \Gamma_0(N))$$

for all $t \in \mathcal{D}_N$ and for all the $\varphi(\gcd(t, N/t))$ inequivalent cusps of $\Gamma_0(N)$ represented by rational numbers of the form $\frac{a}{t} \in \mathcal{S}_N$ with $\gcd(a, t) = 1$.

The index of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$ is given by $\psi(N)$, where ψ is as defined in (2.2) (see [10]). The *valence formula* for $\Gamma_0(N)$ (see [3] or [18]) states:

$$(3.8) \quad \sum_{P \in \Gamma_0(N) \backslash \mathfrak{H}} \frac{1}{n_P} \cdot \mathrm{ord}_P(f) + \sum_{s \in \mathcal{S}_N} \mathrm{ord}_s(f; \Gamma_0(N)) = \frac{k \cdot \psi(N)}{24},$$

where $k \in \mathbb{Z}$, $f : \mathfrak{H} \rightarrow \mathbb{C}$ is a meromorphic function that transforms like a modular forms of weight $k/2$ on $\Gamma_0(N)$ which is also meromorphic at the cusps of $\Gamma_0(N)$ and n_P is the number of elements in the stabilizer of P in the group $\Gamma_0(N)/\{\pm I\}$, where $I \in \mathrm{SL}_2(\mathbb{Z})$ denotes the identity matrix. In particular, if f is an eta quotient, then from (3.8) we obtain

$$(3.9) \quad \sum_{s \in \mathcal{S}_N} \mathrm{ord}_s(f; \Gamma_0(N)) = \frac{k \cdot \psi(N)}{24},$$

because eta quotients do not have poles or zeros on \mathfrak{H} . it follows from (3.9) and from (3.7) that for an eta quotient f of weight $k/2$ on $\Gamma_0(N)$, the valence formula further reduces to

$$(3.10) \quad \sum_{t|N} \varphi(\gcd(t, N/t)) \cdot \mathrm{ord}_{1/t}(f; \Gamma_0(N)) = \frac{k \cdot \psi(N)}{24}.$$

Since $\mathrm{ord}_{1/t}(f; \Gamma_0(N)) \in \frac{1}{24}\mathbb{Z}$ (see (3.5)), from (3.10) it follows that of any particular weight, there are only finitely many holomorphic eta quotients on $\Gamma_0(N)$. More precisely, the number of holomorphic eta quotients of weight $k/2$ on $\Gamma_0(N)$ is at most the number of solutions of the following equation

$$(3.11) \quad \sum_{t \in \mathcal{D}_N} \varphi(\gcd(t, N/t)) \cdot x_t = k \cdot \psi(N)$$

in nonnegative integers x_t .

We define the *order map* $\mathcal{O}_N : \mathbb{Z}^{\mathcal{D}_N} \rightarrow \frac{1}{24}\mathbb{Z}^{\mathcal{D}_N}$ of level N as the map which sends $X \in \mathbb{Z}^{\mathcal{D}_N}$ to the ordered set of orders of the eta quotient η^X at the cusps $\{1/t\}_{t \in \mathcal{D}_N}$ of $\Gamma_0(N)$. Also, we define the *order matrix* $A_N \in \mathbb{Z}^{\mathcal{D}_N \times \mathcal{D}_N}$ of level N by

$$(3.12) \quad A_N(t, d) := 24 \cdot \mathrm{ord}_{1/t}(\eta_d; \Gamma_0(N))$$

for all $t, d \in \mathcal{D}_N$. For example, for a prime power p^n , we have

$$(3.13) \quad A_{p^n} = \begin{pmatrix} p^n & p^{n-1} & p^{n-2} & \cdots & p & 1 \\ p^{n-2} & p^{n-1} & p^{n-2} & \cdots & p & 1 \\ p^{n-4} & p^{n-3} & p^{n-2} & \cdots & p & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & p & p^2 & \cdots & p^{n-1} & p^{n-2} \\ 1 & p & p^2 & \cdots & p^{n-1} & p^n \end{pmatrix}.$$

By linearity of the order map, we have

$$(3.14) \quad \mathcal{O}_N(X) = \frac{1}{24} \cdot A_N X.$$

For $r \in \mathbb{N}$, if $Y, Y' \in \mathbb{Z}^{\mathcal{D}_N^r}$ is such that $Y - Y'$ is nonnegative at each element of \mathcal{D}_N^r , then we write $Y \geq Y'$. In particular, for $X \in \mathbb{Z}^{\mathcal{D}_N}$, the eta quotient η^X is holomorphic if and only if $A_N X \geq 0$.

From (3.12) and (3.5), we note that $A_N(t, d)$ is multiplicative in N, t and d . Hence, it follows that

$$(3.15) \quad A_N = \bigotimes_{\substack{p^n \parallel N \\ p \text{ prime}}} A_{p^n},$$

where by \otimes , we denote the Kronecker product of matrices.*

It is easy to verify that for a prime power p^n , the matrix A_{p^n} is invertible with the tridiagonal inverse:

$$(3.16) \quad A_{p^n}^{-1} = \frac{1}{p^n \cdot (p - \frac{1}{p})} \begin{pmatrix} p & -p & & & 0 \\ -1 & p^2 + 1 & -p^2 & & \\ & -p & p \cdot (p^2 + 1) & -p^3 & \\ & & \ddots & \ddots & \ddots \\ 0 & & & -p^2 & p^2 + 1 & -1 \\ & & & & -p & p \end{pmatrix},$$

where for each positive integer $j < n$, the nonzero entries of the column $A_{p^n}^{-1}(_, p^j)$ are the same as those of the column $A_{p^n}^{-1}(_, p)$ shifted down by $j - 1$ entries and multiplied with $p^{\min\{j-1, n-j-1\}}$. More precisely,

$$(3.17) \quad p^n \cdot (p - \frac{1}{p}) \cdot A_{p^n}^{-1}(p^i, p^j) = \begin{cases} p & \text{if } i = j = 0 \text{ or } i = j = n \\ -p^{\min\{j, n-j\}} & \text{if } |i - j| = 1 \\ p^{\min\{j-1, n-j-1\}} \cdot (p^2 + 1) & \text{if } 0 < i = j < n \\ 0 & \text{otherwise.} \end{cases}$$

For general N , the invertibility of the matrix A_N now follows by (3.15). Hence, any eta quotient on $\Gamma_0(N)$ is uniquely determined by its orders at the set of the cusps $\{1/t\}_{t \in \mathcal{D}_N}$ of $\Gamma_0(N)$. In particular, for distinct $X, X' \in \mathbb{Z}^{\mathcal{D}_N}$, we have $\eta^X \neq \eta^{X'}$. The last statement is also implied by the uniqueness of q -series expansion: Let $\eta^{\hat{X}}$ and $\eta^{\hat{X}'}$ be the *eta products* (i. e. $\hat{X}, \hat{X}' \geq 0$) obtained by multiplying η^X and $\eta^{X'}$ with a common denominator. The claim follows by induction on the weight of $\eta^{\hat{X}}$ (or equivalently, the weight of $\eta^{\hat{X}'}$) when we compare the corresponding first two exponents of q occurring in the q -series expansions of $\eta^{\hat{X}}$ and $\eta^{\hat{X}'}$.

*Kronecker product of matrices is not commutative. However, since any given ordering of the primes dividing N induces a lexicographic ordering on \mathcal{D}_N with which the entries of A_N are indexed, Equation (3.15) makes sense for all possible orderings of the primes dividing N .

4. THE FINITENESS

In this section, we prove the finiteness of irreducible holomorphic eta quotients of a given level (see the corollary of Theorem 1).

Let A_N be the order matrix of level N (see (3.12)). Since all the entries of A_N^{-1} are rational (see (3.15) and (3.16)), for each $t \in \mathcal{D}_N$, there exists a smallest positive integer $m_{t,N}$ such that $m_{t,N} \cdot A_N^{-1}(_, t)$ has integer entries, where $A_N^{-1}(_, t)$ denotes the column of A_N indexed by $t \in \mathcal{D}_N$. We define $B_N \in \mathbb{Z}^{\mathcal{D}_N \times \mathcal{D}_N}$ by

$$(4.1) \quad B_N(_, t) := m_{t,N} \cdot A_N^{-1}(_, t) \text{ for all } t \in \mathcal{D}_N.$$

Clearly, B_N is invertible over \mathbb{Q} . Recall that for $X \in \mathbb{Z}^{\mathcal{D}_N}$, η^X is holomorphic if and only if $A_N X \geq 0$ (see (3.14)). We define the eta quotient F_N by

$$(4.2) \quad F_N := \prod_{t \in \mathcal{D}_N} \eta^{B_N(_, t)}.$$

The lemma below follows trivially:

Lemma 1. *For $N \in \mathbb{N}$, let F_N be as defined above. Then for $X \in \mathbb{Z}^{\mathcal{D}_N}$, both of the eta quotients $f := \eta^X$ and F_N/f are holomorphic if and only if $X \in B_N \cdot [0, 1]^{\mathcal{D}_N}$. \square*

Let $X \in \mathbb{Z}^{\mathcal{D}_N} \setminus \{0\}$ be such that η^X is a holomorphic eta quotient which is not factorizable on $\Gamma_0(N)$. Define $Y \in \mathbb{Z}^{\mathcal{D}_N}$ by $Y := B_N^{-1}X$. Suppose, for some $t \in \mathcal{D}_N$, we have $Y_t \geq 1$. Then η^X is divisible by the nonconstant holomorphic eta quotient $\eta^{B_N(_, t)}$. Since η^X is not factorizable on $\Gamma_0(N)$, we conclude that $X = B_N(_, t)$. Thus, we have proved that

Lemma 2. *For $N \in \mathbb{N}$, let $B_N \in \mathbb{Z}^{\mathcal{D}_N \times \mathcal{D}_N}$ be as defined in (4.1). For $X \in \mathbb{Z}^{\mathcal{D}_N}$, if η^X is a holomorphic eta quotient which is not factorizable on $\Gamma_0(N)$, then either $X \in B_N \cdot [0, 1]^{\mathcal{D}_N}$ or $X = B_N(_, t)$ for some $t \in \mathcal{D}_N$. \square*

Since for $N \in \mathbb{N}$, there are only finitely many lattice points in the bounded polytope $B_N \cdot [0, 1]^{\mathcal{D}_N}$, from Lemma 2 it follows that there are only finitely many holomorphic eta quotients on $\Gamma_0(N)$ which are not factorizable on $\Gamma_0(N)$ (e. g. the irreducible holomorphic eta quotients whose levels divide N).

Proof of Theorem 1.(a). Let f be a holomorphic eta quotient on $\Gamma_0(N)$ which is not factorizable on $\Gamma_0(N)$. From the above lemma, we see that the weight of f is at most equal to the maximum value of $\sigma(X)/2$, where X varies over $B_N \cdot [0, 1]^{\mathcal{D}_N}$ and σ is as defined in (3.2). Since for all $t \in \mathcal{D}_N$, the sum of all the entries in the column $B_N(_, t)$ of B_N is positive (see (4.8)), It follows that

$$\max_{X \in B_N \cdot [0, 1]^{\mathcal{D}_N}} \sigma(X) = \sum_{t \in \mathcal{D}_N} \sigma(B_N(_, t)).$$

Hence, it suffices to show that

$$(4.3) \quad \kappa(N) = \sum_{d \in \mathcal{D}_N} \sigma(B_N(_, d)).$$

Since for $N \in \mathbb{N}$ and $t \in \mathcal{D}_N$, all the entries of the columns $A_N^{-1}(_, t)$ are multiplicative in N and t (see (3.15)), so is the smallest positive integer $m_{t,N}$ such that $m_{t,N} \cdot A_N^{-1}(_, t) \in \mathbb{Z}^{\mathcal{D}_N}$ (see Lemma 4 in [4]). Hence, from the multiplicativity of $A_N^{-1}(d, t)$ in N , d and t (see (3.15)), it follows that $B_N(d, t)$ (see (4.1)) is also multiplicative in N , d and t . That implies:

$$(4.4) \quad B_N = \bigotimes_{\substack{p \in \wp_N \\ p^n \parallel N}} B_{p^n},$$

where \wp_N denotes the set of prime divisors of N . For a prime p , from (4.1) and (3.16), we have

$$(4.5) \quad B_{p^n} = \begin{pmatrix} p & -p & & & & \\ -1 & p^2 + 1 & -p & & & 0 \\ & -p & p^2 + 1 & -p & & \\ & & \ddots & \ddots & \ddots & \\ 0 & & & -p & p^2 + 1 & -1 \\ & & & & -p & p \end{pmatrix}.$$

Summing up the entries of each column of B_{p^n} , we get:

$$(4.6) \quad \sigma(B_{p^n}(_, p^j)) = \begin{cases} p - 1 & \text{if } j = 0 \text{ or } j = n \\ (p - 1)^2 & \text{otherwise.} \end{cases}$$

Since (4.4) implies that

$$(4.7) \quad B_N(_, t) = \bigotimes_{\substack{p \in \wp_N \\ p^j \parallel t}} B_{p^n}(_, p^j) \text{ for all } d \in \mathcal{D}_N,$$

from (4.6) we get:

$$(4.8) \quad \begin{aligned} \sigma(B_N(_, t)) &= \prod_{\substack{p \in \wp_N \\ p^j \parallel t}} \sigma(B_{p^n}(_, p^j)) \\ &= \left(\prod_{\substack{p \in \wp_N \\ p \nmid \gcd(t, N/t)}} (p - 1) \right) \cdot \prod_{\substack{p \in \wp_N \\ p \mid \gcd(t, N/t)}} (p - 1)^2 \\ &= \varphi(\text{rad}(N)) \cdot \varphi(\text{rad}(\gcd(t, N/t))). \end{aligned}$$

Since $\varphi(\text{rad}(\gcd(t, N/t)))$ is multiplicative in $t \in \mathcal{D}_N$, the summatory function $N \mapsto \sum_{t \in \mathcal{D}_N} \varphi(\text{rad}(\gcd(t, N/t)))$ is multiplicative in N . So,

$$\begin{aligned}
 (4.9) \quad \sum_{t \in \mathcal{D}_N} \varphi(\text{rad}(\gcd(t, N/t))) &= \prod_{\substack{p \in \varphi_N \\ p^n \parallel N}} \sum_{j=0}^n \varphi(\text{rad}(p^{\min\{j, n-j\}})) \\
 &= \prod_{\substack{p \in \varphi_N \\ p^n \parallel N}} ((n-1)(p-1) + 2).
 \end{aligned}$$

Now, (4.3) follows from (4.8) and (4.9).

The only $X \in B_N \cdot [0, 1]^{\mathcal{D}_N}$ with $\sigma(X) = \kappa(N)$ is $X = \sum_{t \in \mathcal{D}_N} B_N(_, t)$. Since $N > 1$, it follows trivially from Lemma 2, that for such an X , the holomorphic eta quotient η^X is factorizable on $\Gamma_0(N)$. \square

Proof of Theorem 1.(b). In Lemma 2, we saw that the holomorphic eta quotients which are not factorizable on $\Gamma_0(N)$ correspond either to the columns of B_N or to the set of lattice points in the the fundamental parallelepiped of B_N . Since the number of the columns of B_N is $d(N)$ and since the number of integer points in the fundamental parallelepiped of a nonsingular integer matrix is equal to the volume of the parallelepiped (see Theorem 2 in [2]), we only require to show that the volume of the fundamental parallelepiped of B_N is $\varphi(N)\psi(N)$. By multiplicativity (see (4.4)), it suffices to show that the determinant of B_{p^n} is $p^{2n}(1 - \frac{1}{p^2})$, which follows trivially after transforming B_{p^n} (see (4.5)) to the following matrix through elementary column operations:

$$\begin{pmatrix}
 p & & & & & & & & & & \\
 -1 & p^2 & & & & & & & & & \\
 & -p & p^2 & & & & & & & & \\
 & & \ddots & \ddots & & & & & & & \\
 & & & -p & p^2 & & & & & & \\
 & & & & -p & p^2-1 & -p & & & & \\
 & & & & & p^2 & -p & & & & \\
 0 & & & & & & \ddots & \ddots & & & \\
 & & & & & & & p^2 & -p & & \\
 & & & & & & & & p^2 & -1 & \\
 & & & & & & & & & p^2 & -1 \\
 & & & & & & & & & & p
 \end{pmatrix}$$

and from the fact that for square matrices A and D , we have

$$\det \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det(A) \det(D).$$

Since the fundamental parallelepiped $B_N \cdot [0, 1]^{\mathcal{D}_N}$ contains zero and since $\eta^0 = 1$, the number of integer points in $B_N \cdot [0, 1]^{\mathcal{D}_N}$ which correspond to nonconstant holomorphic eta quotients on $\Gamma_0(N)$ is less than $\varphi(N)\psi(N) + d(N)$. \square

Proof of Theorem 1.(c). For a prime p , from (4.5) we get that the eta quotient $\eta^{B_{p^n}(-, p^j)}$ is of level p^n if and only if $j \geq n - 1$. Hence, from (4.7) it follows that for $N \in \mathbb{N}$, the eta quotient $\eta^{B_N(-, t)}$ is of level N if and only if for each prime divisor p of N , we have $p^{n-1} | t$, where $n \in \mathbb{N}$ is such that $p^n \parallel N$. It is trivial to note that the number of such divisors t of N is $2^{\omega(N)}$, where $\omega(N)$ denotes the number of prime divisors of N . Now, the claim follows from Lemma 2 and from the proof of Theorem 1.(b) above. \square

5. THE COMMON MULTIPLE WITH THE LEAST WEIGHT

In the previous section, we saw that if a holomorphic eta quotient on $\Gamma_0(N)$ is not factorizable on $\Gamma_0(N)$, then its weight is at most equal to $\kappa(N)/2$. In this section, we show that $\kappa(N)/2$ is the smallest possible weight for an eta quotient f such that for each holomorphic eta quotient g which is not factorizable on $\Gamma_0(N)$, f/g is holomorphic (see Theorem 2).

Lemma 3. *For $N \in \mathbb{N}$ and $t \in \mathcal{D}_N$, the holomorphic eta quotient $\eta^{B_N(-, t)}$ is not factorizable on $\Gamma_0(N)$, where $B_N \in \mathbb{Z}^{\mathcal{D}_N \times \mathcal{D}_N}$ is as defined in (4.1).*

Proof. For $t \in \mathcal{D}_N$ and for $Y = A_N \cdot B_N(-, t) \in \mathbb{Z}^{\mathcal{D}_N}$, from (4.1) we get

$$Y_d = \begin{cases} m_{t,N} & \text{if } d = t \\ 0 & \text{otherwise} \end{cases}$$

for all $d \in \mathcal{D}_N$. Recall that for $X \in \mathbb{Z}^{\mathcal{D}_N}$, η^X is holomorphic if and only if $A_N X \geq 0$ (see (3.14)). Suppose, $\eta^{B_N(-, t)}$ is factorizable on $\Gamma_0(N)$. Then there are $X', X'' \in \mathbb{Z}^{\mathcal{D}_N} \setminus \{0\}$ with $B_N(-, t) = X' + X''$ such that $A_N X' \geq 0$ and $A_N X'' \geq 0$. Hence, there exist $m', m'' > 0$ with $m_{t,N} = m' + m''$ such that for $d \in \mathcal{D}_N$, we have

$$(A_N X')_d = \begin{cases} m' & \text{if } d = t, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad (A_N X'')_d = \begin{cases} m'' & \text{if } d = t, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, we have $X' = m' \cdot A_N^{-1}(-, t)$ and $X'' = m'' \cdot A_N^{-1}(-, t)$. Since $m', m'' < m_{t,N}$ and since $m_{t,N}$ is the smallest positive integer such that $m_{t,N} \cdot A_N^{-1}(-, t) \in \mathbb{Z}^{\mathcal{D}_N}$, we conclude that $X' \notin \mathbb{Z}^{\mathcal{D}_N}$ and $X'' \notin \mathbb{Z}^{\mathcal{D}_N}$. Thus, we get a contradiction! Hence, $\eta^{B_N(-, t)}$ is not factorizable on $\Gamma_0(N)$. \square

Proof of Theorem 2. Let F_N be the same as in (4.2). Then Lemma 1 and Lemma 2 together imply that if a holomorphic eta quotient h on $\Gamma_0(N)$ is divisible by F_N , then it is divisible by all the holomorphic eta quotients on $\Gamma_0(N)$ which are not factorizable on $\Gamma_0(N)$.

Conversely, let a holomorphic eta quotient h on $\Gamma_0(N)$ be divisible by each holomorphic eta quotient g on $\Gamma_0(N)$ which is not factorizable on $\Gamma_0(N)$. Let $B_N \in \mathbb{Z}^{\mathcal{D}_N \times \mathcal{D}_N}$ be as defined in (4.1). Then Lemma 3 implies that h is

divisible by $\eta^{B_N(-,t)}$ for all $t \in \mathcal{D}_N$. So in particular, we have

$$(5.1) \quad \text{ord}_{1/t}(h; \Gamma_0(N)) \geq \text{ord}_{1/t}(\eta^{B_N(-,t)}; \Gamma_0(N)) = \text{ord}_{1/t}(F_N; \Gamma_0(N)),$$

where the last equality holds since F_N is the product of all the eta quotients $\{\eta^{B_N(-,t)}\}_{t \in \mathcal{D}_N}$, and since (4.1) and (3.14) together imply that $\eta^{B_N(-,t)}$ has nonzero order only at the cusp $1/t$ of $\Gamma_0(N)$. Since any eta quotient on $\Gamma_0(N)$ is uniquely determined by its orders at the set of the cusps $\{1/t\}_{t \in \mathcal{D}_N}$ of $\Gamma_0(N)$, from (5.1) it follows that h is divisible by F_N . \square

6. EXAMPLES OF IRREDUCIBLE HOLOMORPHIC ETA QUOTIENTS

In this section, we shall show that there exist holomorphic eta quotients of arbitrarily large weights (see Theorem 3).

Lemma 4. *For $N \in \mathbb{N}$ and $t \in \mathcal{D}_{N/\text{rad}(N)}$, the holomorphic eta quotient $\eta^{B_N(-,t)}$ is irreducible, where $B_N \in \mathbb{Z}^{\mathcal{D}_N \times \mathcal{D}_N}$ is as defined in (4.1).*

Proof. Now, recall from Theorem 3 in [4] that a holomorphic eta quotient on $\Gamma_0(N)$ is reducible only if it is factorizable on some $\Gamma_0(M)$ for some multiple M of N with $\text{rad}(M) = \text{rad}(N)$. Suppose, for some $t \in \mathcal{D}_{N/\text{rad}(N)}$ the holomorphic eta quotient $\eta^{B_N(-,t)}$ is reducible. Then there exists a multiple M of N with $\text{rad}(M) = \text{rad}(N)$ such that $\eta^{B_N(-,t)}$ is factorizable on $\Gamma_0(M)$. Since $t \in \mathcal{D}_{N/\text{rad}(N)} \subseteq \mathcal{D}_{M/\text{rad}(M)}$, it follows from (4.7) and (4.5) that $B_M(d, t) = B_N(d, t)$ for all $d|N$ and $B_M(d, t) = 0$ if $d \nmid N$. In other words, we have $\eta^{B_N(-,t)} = \eta^{B_M(-,t)}$ which is not factorizable on $\Gamma_0(M)$ by Lemma 3. Thus, we get a contradiction! Hence, for all $t \in \mathcal{D}_{N/\text{rad}(N)}$, $\eta^{B_N(-,t)}$ is irreducible. \square

Proof of Theorem 3. Since for all $X \in \mathbb{Z}^{\mathcal{D}_N}$, the weight of the eta quotient η^X is $\sigma(X)/2$, the theorem follows immediately from Lemma 4, (4.8) and from the fact that for $t = N/\text{rad}(N)$, the eta quotient $\eta^{B_N(-,t)}$ is of level N (see (4.7) and (4.5)). \square

7. COMPARISON OF THE WEIGHTS

In the tables below, we compare $k_{\max}(N)$ with $\kappa(N)$, where $k_{\max}(N)/2$ is the maximum weight of a holomorphic eta quotient of level N which is not factorizable on $\Gamma_0(N)$, whereas $\kappa(N)/2$ is the weight of the eta quotient F_N which we defined in Theorem 2. From (2.1) and from the discussion about holomorphic eta quotients on $\Gamma_0(p)$ in Section 1, it follows that

$$\kappa(p) = 2(p-1) = 2k_{\max}(p)$$

for each prime p . So, we consider only composite levels in the following tables. In the table on the left, we list prime power levels, whereas the levels listed in the table on the right are products of small primes or products of small powers of such primes.

N	$k_{\max}(N)$	$\kappa(N)$
4	1	3
8	1	4
9	4	8
16	2	5
25	16	24
27	4	12
32	2	6
49	36	48
64	3	7
81	9	16
121	100	120
125	16	40
128	3	8
169	144	168
243	10	20
256	4	9
289	256	288
343	36	84
361	324	360
512	5	10
529	484	528
625	41	56
729	15	24
841	784	840
1024	6	11
1331	100	220
2048	6	12
2187	14	28

N	$k_{\max}(N)$	$\kappa(N)$
6	2	8
10	4	16
12	3	12
14	6	24
15	8	32
18	5	16
20	5	24
21	12	48
22	10	40
24	5	16
26	12	48
28	8	36
30	15	64
34	16	64
35	24	96
38	18	72
39	24	96
46	44	176
51	32	128
55	40	160
69	44	176
77	60	240
85	64	256
87	56	224
91	72	288
95	72	288
115	88	352
119	96	384

ACKNOWLEDGMENTS

I am thankful to Sander Zwegers, who asked during my talk at Cologne whether a Mersmann type finiteness theorem holds if we keep the level of the eta quotients fixed instead of their weight. Corollary 1 is precisely an answer to this question. I would like to thank Don Zagier, Christian Weiß, Armin Straub, Nadim Rustom and Christian Kaiser for their comments. I made most of the computations for the above tables using PARI/GP [21]. I am also very thankful to Danylo Radchenko for his comments as well as for computing $k_{\max}(24)$, $k_{\max}(28)$ and $k_{\max}(30)$. He made these computations using Normaliz [20]. I am grateful to the CIRM : FBK (International Center for Mathematical Research of the Bruno Kessler Foundation) in Trento for providing me with an office space and supporting me with a fellowship during the preparation of this article.

REFERENCES

- [1] S. Ahlgren and K. Ono, “Addition and counting: the arithmetic of partitions,” *Notices Amer. Math. Soc.*, vol. 48, no. 9, pp. 978–984, 2001, MR 1854533, Zbl 1024.11063.

- [2] A. Barvinok, “Lattice points, polyhedra, and complexity,” in *Geometric combinatorics*, ser. IAS/Park City Math. Ser. Amer. Math. Soc., Providence, RI, 2007, vol. 13, pp. 19–62, MR 2383125, Zbl 1144.52017.
- [3] B. C. Berndt, *Ramanujan’s Notebooks. Part III*. Springer-Verlag, New York, 1991, <http://dx.doi.org/10.1007/978-1-4612-0965-2>.
- [4] S. Bhattacharya, “Algorithmic determination of irreducibility of holomorphic eta quotients,” preprint, <http://arxiv.org/pdf/1602.03087.pdf>.
- [5] —, “Finiteness of simple holomorphic eta quotients of a given weight,” preprint, <http://arxiv.org/pdf/1602.02825.pdf>.
- [6] —, “Holomorphic eta quotients of weight $1/2$,” preprint, <http://arxiv.org/pdf/1602.02835.pdf>.
- [7] —, “Infinite families of simple holomorphic eta quotients,” in preparation.
- [8] —, “Factorization of holomorphic eta quotients,” Ph.D thesis, Rheinische Friedrich-Wilhelms-Universität Bonn, 2014, hss.ulb.uni-bonn.de/2014/3711/3711.pdf.
- [9] H. Cohen, *Number Theory, Volume II: Analytic and Modern Tools*. Springer-Verlag, New York, 2007, Graduate Texts in Mathematics. 240, <http://dx.doi.org/10.1007/978-0-387-49894-2>.
- [10] F. Diamond and J. Shurman, *A First Course in Modular Forms*. Springer-Verlag, New York, 2005, Graduate Texts in Mathematics. 228, <http://dx.doi.org/10.1007/b138781>.
- [11] K. Harada, “Moonshine” of finite groups, ser. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2010, MR 2722318, <http://dx.doi.org/10.4171/090>.
- [12] G. Köhler, *Eta products and theta series identities*, ser. Springer Monographs in Mathematics. Springer, Heidelberg, 2011, MR 2766155, <http://dx.doi.org/10.1007/978-3-642-16152-0>.
- [13] Y. Martin, “Multiplicative η -quotients,” *Trans. Amer. Math. Soc.*, vol. 348, no. 12, pp. 4825–4856, 1996, MR 1376550, <http://dx.doi.org/10.1090/S0002-9947-96-01743-6>.
- [14] G. Mersmann, “Holomorphe η -produkte und nichtverschwindende ganze modulformen für $\Gamma_0(N)$,” Diplomarbeit, Rheinische Friedrich-Wilhelms-Universität Bonn, 1991, <https://sites.google.com/site/soumyabhattacharya/miscellany/Mersmann.pdf>.
- [15] D. Mumford, “Varieties defined by quadratic equations,” in *Questions on Algebraic Varieties (C.I.M.E., III Ciclo, Varenna, 1969)*. Edizioni Cremonese, Rome, 1970, pp. 29–100, MR 0282975.
- [16] M. Newman, “Construction and application of a class of modular functions,” *Proc. London. Math. Soc.* (3), vol. 7, pp. 334–350, 1957, MR 0091352.
- [17] —, “Construction and application of a class of modular functions. II,” *Proc. London Math. Soc.* (3), vol. 9, pp. 373–387, 1959, MR 0107629.
- [18] R. A. Rankin, *Modular forms and functions*. Cambridge University Press, Cambridge, 1977, MR 0498390.
- [19] J. Rouse and J. J. Webb, “On spaces of modular forms spanned by eta-quotients,” *Adv. Math.*, vol. 272, pp. 200–224, 2015, MR 3303232, Zbl 1327.11026, <http://dx.doi.org/10.1016/j.aim.2014.12.002>.
- [20] The Normaliz Group, Normaliz version 2.12.02, Osnabrück, 2015, <http://www.home.uni-osnabrueck.de/wbruns/normaliz/>.
- [21] The PARI Group, PARI/GP version 2.7.0, Bordeaux, 2014, <http://pari.math.u-bordeaux.fr/>.
- [22] A. van der Poorten and K. S. Williams, “Values of the Dedekind eta function at quadratic irrationalities,” *Canad. J. Math.*, vol. 51, no. 1, pp. 176–224, 1999, MR 1692895, Zbl 0936.11026, <http://dx.doi.org/10.4153/CJM-1999-011-1>.
- [23] D. Zagier, “Elliptic modular forms and their applications,” in *The 1-2-3 of modular forms*, ser. Universitext. Springer, Berlin, 2008, pp. 1–103, MR 2409678, http://dx.doi.org/10.1007/978-3-540-74119-0_1.